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1984 J. Phys. A: Math. Gen. 17 2453

(http://iopscience.iop.org/0305-4470/17/12/016)

J. Phys. A: Math. Gen. 17 (1984) 2453–2462. Printed in Great Britain

Quantised adiabatic charge transport in the presence of substrate disorder and many-body interaction

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Received 3 April 1984

Abstract. The result for the quantised charge transport induced by an adiabatically varying substrate potential is generalised to the case in which both substrate disorder and many-body interaction are present. The application of our theory to the problem of the integral and fractional quantised Hall effect is discussed.

1. Introduction

In a recent published paper Thouless (1983) considered an electronic system acted on by a periodic substrate potential which varies adiabatically and periodically in time, and discovered that the charge transport induced by such a varying potential is quantised. The same phenomenon was then studied by Zee (1984) using relativistic field theory, and a similar result was obtained. More recently (Simon 1983), this kind of quantisation was analysed in the language of fibre bundles, and was used to classify the topological structures of a class of quantum mechanical systems.

The original theories were general enough to allow both the shapes and the manner of change of the periodical potential to be quite arbitrary except for the following two assumptions. (1) The Fermi level always lies in an energy gap between the filled states and the empty states. In other words, the spectrum for the instantaneous Hamiltonian takes such a form that there is always an energy gap between the first N states and the rest. (2) The eigenstates for the instantaneous Hamiltonian are perfect Bloch waves. Therefore neither substrate disorder nor many-body interaction are allowed. Unlike the first assumption which is necessary for the application of the adiabatic theorem, the second was used only for technical convenience. The present investigation is thus devoted to removing the second assumption, and to generalising the result to the case of substrate disorder and many-body interaction.

The adiabatic theorem will be used for the N-body wavefunctions, and the first assumption in the original theory will then be put into a slightly different form. That is, the ground state for the instantaneous many-body Hamiltonian is always isolated from the rest by a finite energy gap (which will be called the Fermi gap). The second assumption will, however, be replaced by a rather general one: the absence of long-range correlation. Under these conditions the quantisation of the charge transport will be re-established.

0305-4470/84/122453+10$02.25

The potentials (one-body or two-body) will be classified into the relevant part and the irrelevant part. If a part of the potentials can be continuously turned off at any time while keeping the Fermi gap unclosed, then it is called the irrelevant part. The physical basis for this classification is the following. So long as the Fermi gap remains open, the charge transport should be a continuous functional of the potentials. But the allowed values for the charge transport are discrete, therefore it must be a constant and independent of the irrelevant part of the potentials. Further, it does not matter whether the Fermi gap is generated by the periodic substrate potential or the many-body interaction or a combination of both. If it is generated by the periodic potential and unclosed by the continuous turning off of the many-body interaction as well as the substrate disorder, then the latter two are irrelevant to the charge transport. On the other hand, if the Fermi gap is predominantly produced by the many-body interaction, then the substrate potentials can be safely forgotten.

The theory of adiabatic charge transport has an interesting application to the quantised Hall effect. Thus, its generalisation provides a new way to prove the stability of the integral quantisation of the Hall conductance against the perturbation of the many-body interaction and the substrate disorder. In the fractionally quantised Hall effect, the Fermi gap is predominantly opened up by the interaction between electrons, therefore the substrate potentials can be neglected as the current theories on this subject have been saying. The quantised Hall effect problem was critically reviewed by Thouless (1984).

Sections 2–4 are devoted to the detailed analyses which establish the quantisation of the adiabatic charge transport in various cases. After the general formulation of the problem in § 2, we first consider, in § 3, the simpler case in which only single-particle interactions are involved. Then we study in § 4 the more difficult but more interesting case in the presence of the many-body interaction. In § 5 we discuss the application of our theory to the problem of the quantised Hall effect.

2. General formulation

Let us start by considering the Fermi system described by the N-body Hamiltonian

$$H(\tau)=\sum_{i=1}^{N}\left[-\frac{1}{2}\left(\frac{\partial}{\partial x_i}\right)^2+U(x_i,\tau)\right]+\sum_{i>j}^{N}V(x_i-x_j) \tag{2.1}$$

where the substrate potential consists of two parts:

$$U(x_i,\tau)=U_1(x_i)+U_2(x_i,\tau). \tag{2.2}$$

The time-dependent part evolves slowly with time τ and repeats itself in every period T. Note that we have made no postulation about the spacial periodicity of the potentials. To make the problem well defined, we impose upon the wavefunctions of the system the generalised periodic boundary condition

$$\psi(x_1,\ldots,x_i+L,\ldots,x_N)=\mathrm{e}^{\mathrm{i}\alpha L}\psi(x_1,\ldots,x_i,\ldots,x_N). \tag{2.3}$$

The independence of the phase parameter α on the particle index 'i' is required by the total antisymmetry. The dimension of the system is assumed to be 1 in the detailed calculations for convenience, while the extension to higher dimensions can be made easily.

At each instant of time, we can in principle find the set of antisymmetrised (and normalised) eigenstates $|n(\tau, \alpha)\rangle$ of $H(\tau)$ under the boundary condition (2.3) parametrised by α. As mentioned before, we assume that the energy of the ground state $|0(\tau, \alpha)\rangle$ is isolated from the others by a finite gap

$$\Delta(\tau, \alpha, L) \tag{2.4}$$

for every $(\tau, \alpha) \in \{0 \leqslant \tau \leqslant T,\ -\pi \leqslant \alpha L \leqslant \pi\}$ and that the system was initially in the ground state. Then we decompose the total and exact density matrix for the system into the instantaneous one and an extra part:

$$\rho(\tau) = \rho_i(\tau) + \Delta\rho(\tau) \tag{2.5}$$

with

$$\rho_i(\tau) \equiv |0(\tau, \alpha)\rangle\langle 0(\tau, \alpha)|. \tag{2.6}$$

Then the adiabatic theorem (Kato 1950) ensures that $\rho(\tau)$ deviates very little from $\rho_i(\tau)$.

The equation of motion reads

$$[H, \Delta\rho] = [H, \rho_i + \Delta\rho] = i\dot{\rho}_i + i\Delta\dot{\rho} \approx i\dot{\rho}_i \tag{2.7}$$

where ρ_i commutes with the Hamiltonian and $\Delta\dot{\rho}$ has been dropped since it is of higher order than $\partial U/\partial\tau$. Further, we can write

$$\langle 0|\dot{\rho}_i|n\rangle = (d/d\tau)\langle 0|\rho_i|n\rangle - \langle\dot{0}|\rho_i|n\rangle - \langle 0|\rho_i|\dot{n}\rangle = -\langle 0|\dot{n}\rangle = \langle\dot{0}|n\rangle \tag{2.8}$$

and

$$[H, \Delta\rho]_{0n} = (E_0 - E_n)\Delta\rho_{0n}; \tag{2.9}$$

therefore

$$\Delta\rho_{0n} = i\langle\dot{0}|n\rangle/(E_0 - E_n) \qquad (\text{for } n \neq 0). \tag{2.10}$$

The charge transport in a period of time T averaged over space is given by

$$\begin{aligned} C &= \frac{\rho}{L}\int_0^T d\tau\,(1 + \Delta\rho_{00})\cdot j_{00} + \frac{e}{L}\int_0^T d\tau \sum_{n\neq 0}(\Delta\rho_{0n} j_{n0} + j_{0n}\Delta\rho_{n0}) \\ &= \frac{e}{L}\int_0^T d\tau\,(1 + \Delta\rho_{00})\langle 0|-i\partial/\partial x|0\rangle \\ &\quad + \frac{e}{L}\int_0^T d\tau\left(\sum_{n\neq 0}\frac{i\langle\dot{0}|n\rangle}{E_0 - E_n}\langle n|-i\partial/\partial x|0\rangle + \text{CC}\right). \end{aligned} \tag{2.11}$$

At this point we should point out that our system really corresponds to the $\alpha = 0$ case, so that the current in the absence of the time variation of the substrate potential is zero, i.e.

$$\langle 0|-i\,\partial/\partial x|0\rangle_{\alpha=0} = 0 \qquad \left(\partial/\partial x \equiv \sum_{i=1}^{N}\partial/\partial x_i\right). \tag{2.12}$$

Thus

$$C(0) = \frac{e}{L}\int_0^T d\tau\left(\sum_{n\neq 0}\frac{i\langle\dot{0}|n\rangle}{E_0 - E_n}\langle n|-i\,\partial/\partial x|0\rangle + \text{CC}\right)_{\alpha=0}. \tag{2.13}$$

In the $\alpha \neq 0$ case we redefine

$$C(\alpha)=\frac{e}{L}\int_0^T \mathrm{d}\tau\left(\sum_{n\neq 0}\frac{\mathrm{i}\langle\dot{0}|n\rangle}{E_0-E_n}\langle n|-\mathrm{i}\,\partial/\partial x|0\rangle+\mathrm{cc}\right)_\alpha \tag{2.14}$$

where $\{\ldots\}_\alpha$ stresses what kind of boundary condition is imposed on the states inside the parentheses. Later we will show that

$$\begin{aligned}\overline{C(\alpha)}&\equiv\int_{-\pi}^{\pi}\frac{\mathrm{d}\beta}{2\pi}C(\alpha)=\frac{L}{2\pi}\int_{-\pi/L}^{\pi/L}\mathrm{d}\alpha\, C(\alpha)\\&=e\times\text{integer}\qquad(\beta\equiv\alpha L)\end{aligned} \tag{2.15}$$

and that

$$\{C(0)-\overline{C(\alpha)}\}\to 0\qquad\text{as } L\to\infty,\ N\to\infty,\qquad\text{with } N/L=\text{constant}. \tag{2.16}$$

In other words, by introducing $C(\alpha)$ we will be able to explain why $C(0)$ is quantised in the thermodynamic limit. This limit is necessary for the quantisation, as has been illustrated by Thouless (1983).

To establish (2.15) and (2.16), it is easier to work in the basis given by

$$|\phi_n\rangle\equiv\exp[-\mathrm{i}\alpha(x_1+x_2+\ldots+x_N)]|n\rangle \tag{2.17}$$

which satisfies the equation

$$H(\tau,\alpha)|\phi_n\rangle=E_n(\tau,\alpha)|\phi_n\rangle \tag{2.18}$$

and the strict periodic boundary condition, where $H(\tau,\alpha)$ is obtained by replacing $\partial/\partial x_j$ everywhere in the original Hamiltonian by $\partial/\partial x_j+\mathrm{i}\alpha$. Then (2.14) becomes

$$C(\alpha)=\frac{e}{L}\int_0^T \mathrm{d}\tau\left(\sum_{n\neq 0}\frac{\mathrm{i}\langle\dot{\phi}_0|\phi_n\rangle}{E_0-E_n}\langle\phi_n|-\mathrm{i}\,\tilde{\partial}/\partial x|\phi_0\rangle+\mathrm{cc}\right) \tag{2.19}$$

where

$$\tilde{\partial}/\partial x\equiv\sum_{j=1}^{N}(\partial/\partial x_j+\mathrm{i}\alpha). \tag{2.20}$$

From (2.18) we have

$$\langle\phi_n|\partial/\partial\alpha|\phi_0\rangle=(E_0-E_n)^{-1}\langle\phi_n|\partial H(\tau,\alpha)/\partial\alpha|\phi_0\rangle=(E_0-E_n)^{-1}\langle\phi_n|-\mathrm{i}\,\tilde{\partial}/\partial x|\phi_0\rangle \tag{2.21}$$

and therefore

$$\begin{aligned}C(\alpha)&=\frac{e}{L}\int_0^T\mathrm{d}\tau\left(\sum_{n\neq 0}\mathrm{i}\langle\dot{\phi}_0|\phi_n\rangle\langle\phi_n|\partial/\partial\alpha|\phi_0\rangle+\mathrm{cc}\right)\\&=\frac{\mathrm{i}e}{L}\int_0^T\mathrm{d}\tau\left(\left\langle\frac{\partial\phi_0}{\partial\tau}\middle|\frac{\partial\phi_0}{\partial\alpha}\right\rangle-\left\langle\frac{\partial\phi_0}{\partial\alpha}\middle|\frac{\partial\phi_0}{\partial\tau}\right\rangle\right).\end{aligned} \tag{2.22}$$

The justification of the last equality is given in appendix 1. Thus

$$\overline{C(\alpha)}=e\,\mathrm{i}\int_{-\pi}^{\pi}\frac{\mathrm{d}\beta}{2\pi}\int_0^T\mathrm{d}\tau\left(\left\langle\frac{\partial\phi_0}{\partial\tau}\middle|\frac{\partial\phi_0}{\partial\beta}\right\rangle-\left\langle\frac{\partial\phi_0}{\partial\beta}\middle|\frac{\partial\phi_0}{\partial\tau}\right\rangle\right). \tag{2.23}$$

Just as has been argued by Thouless *et al* (1982), such an expression must be an integer times the electron charge e. The justification of (2.16) will be given in §§ 3 and 4.

3. Interaction only with substrate

In the absence of many-body interaction, the states in (2.14) may be written as determinants of single-particle states. The element $\langle n|-\mathrm{i}\,\partial/\partial x|0\rangle$ vanishes unless $|n\rangle$ differs from $|0\rangle$ only by a single-particle excitation, say the pth single-particle state in $|0\rangle$ being excited to the qth outside $|0\rangle$. Then the formula for $C(\alpha)$ can be turned into a form only involving single-particle quantities:

$$C(\alpha)=\frac{e}{L}\int_0^T \mathrm{d}\tau\left(\sum_{p<f}\sum_{q>f}\frac{\mathrm{i}\langle \dot{p}|q\rangle}{\varepsilon_p-\varepsilon_q}\langle q|-\mathrm{i}\,\partial/\partial x|p\rangle+\mathrm{cc}\right) \tag{3.1}$$

where, for example, $-\mathrm{i}\,\partial/\partial x$ now stands for the single-particle momentum operator. The first summation is taken over all single-particle states below the Fermi gap, while the second is over those above the Fermi gap. After some manipulation this can be written in terms of the single-particle Green functions:

$$C(\alpha)=-\frac{\mathrm{e}}{L}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\int_0^T \mathrm{d}\tau\,\mathrm{Tr}\left(g\frac{\partial}{\partial x}gg\dot{h}\right) \tag{3.2}$$

where

$$h\equiv -\tfrac{1}{2}(\mathrm{d}/\mathrm{d}x)^2+U(x,\tau),$$
$$\dot{h}=(\partial/\partial\tau)U(x,\tau),\qquad g=(z-h)^{-1}. \tag{3.3}$$

The integration contour encloses the filled states energies below the Fermi gap. If we make use of the relation

$$\partial/\partial x=[x,h]=[g^{-1},x] \tag{3.4}$$

then

$$\mathrm{Tr}\left[g\frac{\partial}{\partial x}gg\dot{h}\right]=\mathrm{Tr}[xgg\dot{h}]-\mathrm{Tr}[gxg\dot{h}]=-\frac{\partial}{\partial z}\mathrm{Tr}[xg\dot{h}]-\frac{\partial}{\partial\tau}\mathrm{Tr}[xg] \tag{3.5}$$

which gives a zero result for $C(\alpha)$, owing to the periodicity in z and τ of the quantities like xg and $xg\dot{h}$. This is certainly nonsense! The reason is that we cannot have such a relation as

$$z-h=g^{-1} \tag{3.6}$$

when the operator x is involved, which takes states satisfying the boundary condition (2.17) to states which do not. The correct way of doing this is to use the related proper operator $\xi(x)$ defined by

$$\xi(x)\equiv x-L\eta(x),\qquad \eta(x)\equiv\begin{cases}\frac{1}{2}, & x>0,\\ -\frac{1}{2}, & x<0.\end{cases} \tag{3.7}$$

Then

$$\partial/\partial x=[g^{-1},\xi]+Lj(0), \tag{3.8}$$
$$j(0)\equiv\tfrac{1}{2}((\partial/\partial x)\delta(x)+\delta(x)\,\partial/\partial x), \tag{3.9}$$

which gives the correct non-zero result for the charge transport:

$$C(\alpha) = -e\oint \frac{d\tau}{2\pi i}\int_0^T d\tau\, \mathrm{Tr}[gj(0)gg\dot{h}]$$

$$= -e\oint \frac{dz}{2\pi i}\int_0^T d\tau \int_{-L/2}^{L/2}\!\!\int\!\!\int dx\, dx'\, dx''[g(x', x)j(0)g(x, x'')g(x'', x')\dot{h}(x')]. \tag{3.10}$$

When z is away from the spectrum, the Green function $g(x, x')$ falls off exponentially when x' is far away from the central peak point and the boundary peak points defined respectively by

$$|x - x'| = 0 \qquad \text{and} \qquad |x - x'| = L, \tag{3.11}$$

where the latter equation can only be satisfied when x and x' sit on the opposite boundaries of the integration range. While the central peak is a general feature of the Green functions whose energy parameter is away from the spectrum, the boundary peaks only arise from the boundary condition that has to be satisfied by the Green functions.

Now, because $j(0)$ contains the δ function centred at $x = 0$, the integrand is appreciable only when

$$x'' \approx x' \approx x = 0. \tag{3.12}$$

Also, the boundary peak condition cannot be satisfied by any one of the three Green functions while still giving appreciable overlapping. Since only the boundary peak parts depend on the boundary condition, the α-dependent part of $C(\alpha)$ must be exponentially small. In conclusion, we must have

$$|C(0) - \overline{C(\alpha)}| \to 0 \tag{3.13}$$

in the thermodynamic limit.

In the above analysis only the existence of a finite Fermi gap has been assumed. In general, we need the main part of the substrate potential to be periodic in space to produce such a gap, with perhaps a weak disorder which does not close it. In this case, the disorder is irrelevant to the charge transport no matter whether it is static or varies periodically in time.

4. With interaction between electrons

In this section we consider the case in which many-body interaction is also present. Our main theme is to prove the vanishing of $(d/d\beta)C(\alpha)$ in the thermodynamic limit, which, with the intermediate value theorem, leads to the conclusion stated in (2.16). This process is done by converting $dC/d\beta$ into the form of correlation functions of local operators centred in macroscopically separated regions. Since N-body quantities are used repeatedly in this section, we list below a set of abbreviations for convenience:

$$x \equiv \sum_{j=1}^{N} x_j, \qquad \delta(x - X) \equiv \sum_{j=1}^{N} \delta(x_j - X), \qquad \frac{\tilde{\partial}}{\partial x} \equiv \sum_{j=1}^{N} \frac{\tilde{\partial}}{\partial x_j} \equiv \sum_{j=1}^{N}\left(\frac{\partial}{\partial x_j} + i\alpha\right),$$

$$J(X)\equiv\tfrac{1}{2}\left(\delta(x-X)\frac{\tilde{\partial}}{\partial x}+\frac{\tilde{\partial}}{\partial x}\delta(x-X)\right)\equiv\tfrac{1}{2}\sum_{i=1}^{N}\left(\delta(x_i-X)\frac{\tilde{\partial}}{\partial x_i}+\frac{\tilde{\partial}}{\partial x_i}\delta(x_i-X)\right). \tag{4.1}$$

Formula (2.20) can be written as

$$C(\alpha)=-\frac{e}{L}\int_0^T \mathrm{d}\tau\oint\frac{\mathrm{d}\xi}{2\pi_{\mathrm{i}}}\mathrm{Tr}\left(G\frac{\tilde{\partial}}{\partial x}GG\dot{H}\right) \tag{4.2}$$

where the trace is taken over all N-fermion states, and the z integration goes around the ground state energy of the system. The N-body Green function is defined by

$$G=(z-H(\tau,\alpha))^{-1} \tag{4.3}$$

and the time derivative of the Hamiltonian is actually

$$\sum_{i=1}^{N}\frac{\partial}{\partial\tau}U(x_i,\tau) \tag{4.4}$$

for only the substrate potential is assumed to be varying in time. Note that

$$\tilde{\partial}/\partial x=[x,H(\tau,\alpha)]=[G^{-1},\xi]+LJ(X_1) \tag{4.5}$$

where ξ is a sum of saw-tooth functions with discontinuities at $x_i=X_1$. Substitution of this relation into (4.2) gives

$$\mathrm{Tr}\left(G\frac{\tilde{\partial}}{\partial x}GG\dot{H}\right)=\mathrm{Tr}(\xi GG\dot{H})-\mathrm{Tr}(G\xi G\dot{H})+L\,\mathrm{Tr}[GJ(X_1)GG\dot{H}] \tag{4.6}$$

where the first two terms can in turn be turned into total derivatives with respect to z or τ, and therefore they become zero after the z or τ integration. Therefore

$$C(\alpha)=-e\int_0^T \mathrm{d}\tau\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\mathrm{Tr}[GJ(X_1)GG\dot{H}]. \tag{4.7}$$

The derivative of this quantity with respect to the phase $\beta\equiv\alpha L$ is readily evaluated by using the following equations:

$$\partial J(X_1)/\partial\alpha=\mathrm{i}\delta(x-X_1),\qquad \partial G/\partial\alpha=\mathrm{i}G\,\tilde{\partial}G/\partial x. \tag{4.8}$$

After applying a similar relation to (4.5) (with the discontinuity of the saw-tooth functions at $x_i=X_2$) to the resulting $\partial C/\partial\beta$, we come up with a large number of happy cancellations, being left with a comparatively simple result:

$$\frac{\partial C}{\partial\beta}=\oint\frac{\mathrm{d}z}{2\pi}\int_0^T \mathrm{d}\tau\,\mathrm{Tr}\{[J(X_2),GJ(X_1)G]G\dot{H}G\}. \tag{4.9}$$

It is easier to work in the second quantisation frame, so we take the trouble (appendix 2) to express everything in terms of some expectation values in the ground state as

$$\begin{aligned}\frac{\partial}{\partial\beta}C=\oint\frac{\mathrm{d}z}{2\pi}\int_0^T \mathrm{d}\tau\, e\{&\langle GJ(X_1)GJ(X_2)G\dot{H}G\rangle-\langle G\dot{H}GJ(X_2)GJ(X_1)G\rangle\\&+\langle J(X_2)G\dot{H}GGJ(X_1)G\rangle-\langle GJ(X_1)GG\dot{H}GJ(X_2)\rangle\\&+\langle G\dot{H}GGJ(X_1)GJ(X_2)\rangle\\&-\langle J(X_2)GJ(X_1)GG\dot{H}G\rangle+\langle GJ(X_1)G\rangle\langle[G\dot{H}G,J(X_2)]\rangle\\&+\langle J(X_2)\rangle\langle[GJ(X_1)G,G\dot{H}G]\rangle+\langle G\dot{H}G\rangle\langle[J(X_2),GJ(X_1)G]\rangle\}.\end{aligned} \tag{4.10}$$

At this stage all the operators can be directly turned into their second quantisation form, while $|\ \rangle$ contains N electrons. To evaluate (4.10), we express G by its Fourier transform. The first pair of terms in the braces in (4.10), for instance, can be written as

$$\int_{C_1}\frac{dz}{2\pi}\int_0^\infty\int\int\int(-i)^4\,d^4t\,\exp[i(t+t'+t''+t''')(z-E_0)][\langle J(X_1,t)J(X_2)\dot{H}(-t')\rangle$$
$$-\langle\dot{H}(-t')J(X_2)J(X_1,t)\rangle+(t\to-t,\,t'\to-t')] \tag{4.11}$$

where C_1 is the upper-half contour surrounding $z=E_0$. Since z is away from the spectrum, the t integrations are effectively cut off to some finite upper bounds. Within these bounds, $J(X_1,t)$ remains a local operator, and becomes negligible as it acts on the region outside a neighbourhood centred at X_1. We call such a neighbourhood the effective domain of the operator. In the thermodynamic limit we can choose X_1 and X_2 to be macroscopically separated. Then we divide $\dot{H}(-t')$ into $\dot{H}_1$ and $\dot{H}_2$, so that the effective domain of $\dot{H}_1$ is macroscopically separated from that of $J(X_2)$, and the effective domain of $\dot{H}_2$ from that of $J(X_1,t)$. If the correlation between operators whose effective domains are separated far away is negligible, then it is valid to make the following approximation:

$$\begin{aligned}
&\langle J(X_1,t)J(X_2)\dot{H}(-t')\rangle-\langle\dot{H}(-t')J(X_2)J(X_1,t)\rangle\\
&\quad=\langle J(X_1,t)J(X_2)\dot{H}_1\rangle+\langle J(X_1,t)J(X_2)\dot{H}_2\rangle-\langle\dot{H}_1J(X_2)J(X_1t)\rangle\\
&\qquad-\langle\dot{H}_2J(X_2)J(X_1,t)\rangle\\
&\quad\simeq\langle J(X,t)\dot{H}_1\rangle\langle J(X_2)\rangle+\langle J(X_2)\dot{H}_2\rangle\langle J(X_1,t)\rangle-\langle\dot{H}_1J(X_1,t)\rangle\langle J(X_2)\rangle\\
&\qquad-\langle\dot{H}_2J(X_2)\rangle\langle J(X_1,t)\rangle\\
&\quad=\langle[J(X_1t),\dot{H}_1]\rangle\langle J(X_2)\rangle+\langle[J(X_2),\dot{H}_2]\rangle\langle J(X_1t)\rangle\\
&\quad\simeq\langle[J(X_1t),\dot{H}(-t')]\rangle\langle J(X_2)\rangle+\langle[J(X_2),\dot{H}(-t')]\rangle\langle J(X_1,t)\rangle
\end{aligned}\tag{4.12}$$

where the error made will go to zero in the thermodynamic limit.

Upon completion of the t integrals (4.11) becomes

$$\oint\frac{dz}{2\pi}\{\langle[GJ(X_1)G,G\dot{H}G]\rangle\langle J(X_2)\rangle+\langle[J(X_2),G\dot{H}G]\rangle\langle GJ(X_1)G\rangle\}. \tag{4.13}$$

After a similar manipulation of the second and third pair of terms in (4.10), we arrive at the rather simplified result

$$\frac{\partial C}{\partial\beta}\simeq e\oint\frac{dz}{2\pi}\int_0^T d\tau\,\langle G\dot{H}G\rangle\langle[J(X_2),GJ(X_1)G]\rangle \tag{4.14}$$

where a large number of cancellations have been accomplished. In fact, with exactly the same argument as made above, even this remaining term must also go to zero in the thermodynamic limit. In this way, we have justified (2.16) and concluded the quantisation of charge transport in the presence of many-body interaction.

5. Application to the quantised Hall effect

We consider a two-dimensional interacting electron system acted on by a magnetic field in the z direction which is perpendicular to the plane, and by an electric field

pointing in the y direction. The existence of a substrate potential is also assumed. The Hall current can be calculated in the moving frame in which E is transformed to zero. The velocity of the frame is simply

$$\boldsymbol{v} = -(E/B)\boldsymbol{e}_x \tag{5.1}$$

while the Hamiltonian in this frame is

$$H(\tau) = \sum_i^N \left[\frac{1}{2}\left(-\mathrm{i}\frac{\partial}{\partial x_i} - eBy_i \right)^2 + \frac{1}{2}\left(-\mathrm{i}\frac{\partial}{\partial y_i} \right)^2 + U(\boldsymbol{r}_i - \boldsymbol{v}\tau) \right] + \sum_{i<j}^N V(\boldsymbol{r}_i - \boldsymbol{r}_j). \tag{5.2}$$

Since the Hall conductance is calculated in the limit of $E = 0$, the time variation of H(through $u(\boldsymbol{r}_i - \boldsymbol{v}t)$) can be considered as adiabatic. The only difference between (5.2) and (2.1) is that we now have $(-\mathrm{i}\,\partial/\partial x_i - eBy_i)$ in place of $-\mathrm{i}\,\partial/\partial x_i$ as in (2.1), but the current operator is also modified correspondingly. One can show easily that the derivations in the previous sections keep valid under these modifications, so the current must continue to be quantised if the Fermi gap is open.

In the first case in which the Fermi gap lies between the main Landau levels (which may be modified by the substrate potential and the many-body interaction), only the magnetic field is relevant to the problem, giving a Hall conductance of

$$\sigma = n\,e^2/h \tag{5.3}$$

where n is the number of filled Landau levels.

In the second place, if the Fermi gap is between two sub-bands split from a Landau level by the periodic part of the substrate potential, then we can ignore the substrate disorder and the many-body interaction. As has been done by Thouless *et al* (1982), the Hall conductance is still quantised in the form of (5.3).

Finally, in the fractional quantised Hall effect, the Fermi gap is believed to be generated by the interaction between electrons, then we can totally neglect the substrate potential. With this idealisation the current in the moving frame is identically zero. This gives a Hall conductance

$$\sigma = \nu\,e^2/h \tag{5.4}$$

with ν being the filling factor of the lowest Landau level.

Acknowledgment

This work was supported by the National Science Foundation, under grant number DMR79-20795, and by the Royal Society.

Appendix 1

To obtain the right-hand side of the last equality in (2.23), we have added a term of

$$\frac{\mathrm{i}e}{L}\int_0^T \mathrm{d}\tau(\langle\dot{\phi}_0|\phi_0\rangle\langle\phi_0|\partial/\partial\alpha|\phi_0\rangle - \mathrm{cc}) \tag{A1}$$

to the left-hand side. Since $|\phi_0\rangle$ is normalised, both $\langle\dot{\phi}_0|\phi_0\rangle$ and $\langle\phi_0|\partial/\partial\alpha|\phi_0\rangle$ must be purely imaginary. As a result, the integrand in the above expression is identically zero.

Appendix 2

By expanding the trace,(4.9), we obtain something like

$$\frac{\partial C}{\partial \beta} = e \oint \frac{dz}{2\pi} \int_0^T d\tau \sum_{m,n,l} \frac{1}{(z-E_m)^2} \frac{1}{z-E_n} \frac{1}{z-E_l} \{\text{independent of } z\}. \tag{A2.1}$$

Because of the z integration, only those terms with at least one of the indices in the summation being equal to 0 (the index for the ground state) survive. After some tedious rearrangement of the terms we obtain (4.10).

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